

# On Jordan type bounds for finite groups of diffeomorphisms of 3-manifolds and Euclidean spaces

Bruno P. Zimmermann

Università degli Studi di Trieste  
Dipartimento di Matematica e Geoscienze  
34127 Trieste, Italy

*Abstract. By a classical result of Jordan, each finite subgroup  $G$  of  $\mathrm{GL}_n(\mathbb{C})$  has an abelian subgroup whose index in  $G$  is bounded by a constant depending only on  $n$ . We consider the problem if this remains true for finite subgroups  $G$  of the diffeomorphism group of a smooth manifold, and show that it is true for all compact 3-manifolds as well as for Euclidean spaces  $\mathbb{R}^n$ ,  $n \leq 6$ . The question remains open at present e.g. for odd-dimensional spheres  $S^n$ ,  $n \geq 5$  and for Euclidean spaces  $\mathbb{R}^n$ ,  $n \geq 7$ .*

## 1. Introduction

By a classical result of Jordan, each finite subgroup  $G$  of  $\mathrm{GL}_n(\mathbb{C})$  has an abelian subgroup  $A$  whose index in  $G$  is bounded by a constant depending only on  $n$  (see [C] for the optimal bound for each  $n$ ). Recently there has been much interest in generalizations, replacing  $\mathrm{GL}_n(\mathbb{C})$  by more general geometrically interesting groups such as diffeomorphism groups of smooth manifolds ([MR1,2],[P1]), automorphism groups of algebraic varieties and the Cremona groups of birational self-maps of the affine  $n$ -dimensional space (cf. [P2], [Se, Theoreme 3.1]).

Following [P1,2], we say that a group  $E$  is a *Jordan group* or has the *Jordan property* if there exists a constant such that every finite subgroup  $G$  of  $E$  has an abelian subgroup of index bounded by this constant. For a smooth manifold  $M$ , let  $\mathrm{Diff}(M)$  denote its diffeomorphism group. The present paper is motivated by the following general:

Question: For which (classes of) smooth manifolds  $M$  is  $\mathrm{Diff}(M)$  a Jordan group?

Whereas this is in general not true for non-compact manifolds ([P1]), it has been conjectured that it is true for compact manifolds (see [MR1,2]); however, it should be true e.g. also for  $\mathrm{Diff}(\mathbb{R}^n)$ .

Note that a Jordan group contains only finitely many finite non-abelian simple subgroups, up to isomorphism; in this regard, it has been shown in [GZ] that  $\mathrm{Diff}(S^n)$  contains only finitely many finite non-abelian simple subgroups, up to isomorphism (and, more generally, for any closed homology  $n$ -sphere, see also [Z1]). It has been

shown in [MR1] that  $\text{Diff}(M)$  is a Jordan group if  $M$  is a compact manifold without odd cohomology; in particular,  $\text{Diff}(S^n)$  is a Jordan group for even dimensions  $n$ , but this remains open for odd dimensions  $n \geq 5$ .

On the basis of the geometrization of 3-manifolds and results of Kojima [K] and the author [Z2], in our first main result we consider the case of compact 3-manifolds:

**Theorem 1.**  *$\text{Diff}(M)$  is a Jordan group for compact 3-manifolds  $M$ .*

In dimension three, this leaves open the question for non-compact 3-manifolds. Concerning dimension four, it is shown in [P2] that there are noncompact, simply connected, smooth 4-manifolds  $M$  such that  $\text{Diff}(M)$  is not a Jordan group. On the other hand, it is shown in [MR2] that, for compact smooth 4-manifolds  $M$  with non-zero Euler characteristic,  $\text{Diff}(M)$  is a Jordan group. The case of the 4-sphere  $S^4$  is considered in [MeZ1,2] where it is shown that, up to 2-fold extension in the case of solvable groups, any finite group with an orientation-preserving smooth action on  $S^4$  (or on any homology 4-sphere) is isomorphic to a subgroup of the orthogonal group  $\text{SO}(5)$ , presenting also a short list of such groups.

Next we consider Euclidean spaces  $\mathbb{R}^n$ . The following is proved in [GMZ]:

**Theorem 2.** ([GMZ]) *Let  $G$  be a finite subgroup of  $\text{Diff}(\mathbb{R}^n)$  (or of  $\text{Diff}(M)$ , for any acyclic  $n$ -manifold  $M$ ). Suppose that  $n \leq 4$ ; then  $G$  is isomorphic to a subgroup of the orthogonal group  $\text{O}(n)$ . In particular, the classical Jordan bound applies to  $G$ , so  $\text{Diff}(\mathbb{R}^n)$  is a Jordan group for  $n \leq 4$ .*

In [GMZ] the case of finite groups of diffeomorphisms of  $\mathbb{R}^5$  is also considered; the classification in this case is not complete but the results imply easily that also  $\text{Diff}(\mathbb{R}^5)$  is a Jordan group (more generally, the results in [GMZ] apply to arbitrary acyclic manifolds).

A main tool for the proof of our second main result is a recent group-theoretical result of Mundet i Riera and Turull [MT] (on the basis of the classification of the finite simple groups).

**Theorem 3.**  *$\text{Diff}(\mathbb{R}^5)$  and  $\text{Diff}(\mathbb{R}^6)$  are Jordan groups (and, more generally,  $\text{Diff}(M)$  for any acyclic 5- or 6-manifold  $M$ ).*

We will present the proof of Theorem 3 for the new case  $n = 6$ ; the same proof works also for  $n = 5$  where it is, in fact, easier. As noted above, the proof for  $n = 6$  uses the full classification of the finite simple groups; the proof for  $n = 5$  instead requires "only" a smaller part of the classification of the finite simple groups (the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most four), see [GMZ], [Z1].

Two interesting cases where the Jordan property is not known at present are those of  $\text{Diff}(S^5)$  and  $\text{Diff}(\mathbb{R}^7)$ . However, it seems likely that  $\text{Diff}(S^n)$  and  $\text{Diff}(\mathbb{R}^n)$  are Jordan groups for all values of  $n$ .

## 2. Proof of Theorem 1

It is easy to see that, if  $\tilde{M}$  is a finite covering of  $M$  such that  $\text{Diff}(\tilde{M})$  is a Jordan group then also  $\text{Diff}(M)$  is a Jordan group. So it is sufficient to consider the case of orientable manifolds, and also of orientation-preserving finite group actions  $G$  (passing eventually to a subgroup of index two of  $G$ ). Also, it is sufficient to consider the case of closed manifolds since, for a compact manifold  $M$  with non-empty boundary, one can reduce to the closed case by taking the double of  $M$  along the boundary, doubling also a given finite group action on  $M$ .

So let  $M$  be a closed orientable 3-manifold and  $G$  a finite group of orientation-preserving diffeomorphisms of  $M$ . If  $\pi_1(M)$  is finite then, by the geometrization of 3-manifolds after Perelman,  $M$  is a spherical 3-manifold and finitely covered by  $S^3$ ; also, any finite group of diffeomorphisms of  $M$  is conjugate to a linear (orthogonal) action. By the classical Jordan bound for linear groups,  $\text{Diff}(S^3)$  is a Jordan group, and hence also  $\text{Diff}(M)$  is a Jordan group.

Assume next that  $M$  is irreducible and has infinite fundamental group; again by the geometrization of 3-manifolds, we can assume that  $M$  is a geometric. Then, if  $M$  does not admit a circle action, by [K, Theorem 4.1] there is a bound on the order of finite subgroups of  $\text{Diff}(M)$  and we are done.

Suppose that  $M$  has a circle action and infinite fundamental group. Then  $M$  is a Seifert fiber space, and by the geometrization of finite group actions on Seifert fiber spaces ([MS]), we can assume that the action of the finite group  $G$  of diffeomorphisms of  $M$  is geometric, and in particular fiber-preserving and normalizing the  $S^1$ -action of  $M$ . Considering a suitable finite covering of  $M$ , we can moreover assume that  $M$  has no exceptional fibers, and hence that the base space of the Seifert fibration (the quotient of the  $S^1$ -action) is a closed orientable surface  $B$  without cone points. The finite group  $G$  projects to a finite group  $\bar{G}$  of diffeomorphisms of the base-surface  $B$ , and we can again assume that  $\bar{G}$  is orientation-preserving.

If  $B$  is a hyperbolic surface (of genus  $g \geq 2$ ) then, by the formula of Riemann-Hurwitz, the order of the finite group  $\bar{G}$  of diffeomorphisms of  $B$  is bounded, and hence  $G$  has a finite cyclic subgroup of bounded index (the intersection of  $G$  with the  $S^1$ -action).

If  $B$  is a torus  $T^2$  then there are two cases. First,  $M$  may be a 3-dimensional torus  $T^3$ ; this acts by rotations on itself. Since the action of  $G$  is geometric, the subgroup  $G_0$  of  $G$  acting trivially on the fundamental group is a subgroup of the  $T^3$ -action and hence abelian of rank at most three (see [Sc] for the geometries of 3-manifolds and their

isometry groups). The factor group  $G/G_0$  acts faithfully on the fundamental group  $\mathbb{Z}^3$  of the 3-torus and is isomorphic to a subgroup of  $\mathrm{GL}_3(\mathbb{Z})$ . Since, by a well-known result of Minkowski, there is a bound on the finite subgroups of  $\mathrm{GL}_n(\mathbb{Z})$  for each  $n$ ,  $G$  has an abelian subgroup  $G_0$  of bounded index.

If  $M$  fibers over  $T^2$  but is not a 3-torus then it belongs to the nilpotent geometry Nil given by the Heisenberg group (see again [Sc]). Now the subgroup  $G_0$  of  $G$  acting trivially on the fundamental group, up to inner automorphisms, is a cyclic subgroup of the  $S^1$  action on  $M$ , and  $G/G_0$  injects into the outer automorphism group  $\mathrm{Out}(\pi_1 M)$  of the fundamental group. The fundamental group of  $M$  has a presentation

$$\pi_1 M = \langle a, b, t \mid [a, b] = t^k, [a, t] = [b, t] = 1 \rangle,$$

with  $k \neq 0$ . Now an easy calculation shows that the subgroup of the outer automorphism group of  $\pi_1 M$  inducing the identity of the factor group  $\pi_1 M / \langle t \rangle \cong \mathbb{Z}^2$  is finite. Since the orders of finite subgroups of  $\mathrm{GL}_2(\mathbb{Z})$  are also bounded,  $G$  has a finite cyclic subgroup  $G_0$  of bounded index.

Finally, if the base-surface is the 2-sphere then either  $M$  has finite fundamental group and is a spherical manifold, or homeomorphic to  $S^2 \times S^1$  (and hence non-irreducible). We note that  $S^2 \times S^1$  belongs to the  $(S^2 \times \mathbb{R})$ -geometry, one of Thurston's eight 3-dimensional geometries; this is the easiest of the eight geometries and can be easily handled, see [Sc] for the isometry group of this geometry.

Summarizing, we have shown that for any closed irreducible 3-manifold  $M$  (and also for  $S^2 \times S^1$ ),  $\mathrm{Diff}(M)$  is a Jordan group.

Suppose that  $M$  is non-irreducible but not  $S^2 \times S^1$ . If  $M$  has a summand other than lens spaces and  $S^2 \times S^1$  then, by [K, Theorem 4.2], the orders of finite diffeomorphism groups of  $M$  are again bounded and we are done.

Suppose next that  $M$  is a connected sum  $\sharp_g(S^2 \times S^1)$  of  $g$  copies of  $S^2 \times S^1$ , with  $g > 1$ . By [Z2],  $G$  has a finite cyclic normal subgroup (the subgroup acting trivially on the fundamental group, up to inner automorphisms) such that the order of the factor group is bounded by a polynomial which is quadratic in  $g$ , so we are done also in this case. Finally, if  $M$  is a connected sum of lens spaces, including  $S^2 \times S^1$ , then  $M$  has a finite covering by a 3-manifold of type  $\tilde{M} = \sharp_g(S^2 \times S^1)$  as before. Now  $\mathrm{Diff}(\tilde{M})$  is a Jordan group and hence also  $\mathrm{Diff}(M)$ .

We have considered all possibilities for  $M$  and completed the proof of Theorem 1.

### 3. Proof of Theorem 3

We prove the theorem for  $n = 6$ ; for  $n = 5$  the theorem follows from [GMZ, Theorem 3], and also a shorter version of the following proof for  $n = 6$  applies.

We want to show that  $\text{Diff}(\mathbb{R}^6)$  is a Jordan group, i.e. that there is a constant such that every finite subgroup  $G$  of  $\text{Diff}(\mathbb{R}^6)$  has an abelian subgroup of index bounded by this constant. By the main result of [MT], if this is true for all finite subgroups  $G$  of  $\text{Diff}(\mathbb{R}^6)$  which are a semidirect product  $G = P \rtimes Q$ , for a finite normal  $p$ -group  $P$  and a finite  $q$ -group  $Q$ , with distinct primes  $p$  and  $q$ , then it is true for all finite subgroups  $G$  of  $\text{Diff}(\mathbb{R}^6)$  (this uses the classification of the finite simple groups). So we have to consider only groups of type  $G = P \rtimes Q$ : given such a group, we have to find an abelian subgroup  $A$  of  $G$  whose index is bounded by a constant not depending on the specific group.

Let  $G = P \rtimes Q$  be as before; we can assume that the action of  $G$  is orientation-preserving. By general Smith fixed point theory, a finite  $q$ -group acting on  $\mathbb{R}^n$  (or on any acyclic  $n$ -manifold) has non-empty fixed point set (see [B], [GMZ, section 2]). So  $Q$  has a global fixed point and is isomorphic to a subgroup of the orthogonal group  $\text{SO}(6)$  (considering the induced linear action on the tangent space of a global fixed point). Hence, by the classical Jordan bound for linear groups, we may assume that  $Q$  is an abelian  $q$ -group.

Let  $F$  denote the fixed point set of  $P$ ; since  $P$  is normal,  $F$  is invariant under the action of  $Q$  and, since the action is orientation-preserving,  $F$  is a submanifold of dimension at most four (i.e., of codimension at least two).

Suppose first that  $F$  has dimension four. Then  $P$  acts as a group of rotations around its fixed point set  $F$  and hence is a cyclic group (isomorphic to a subgroup of  $\text{SO}(2)$ ). By conjugation, every element of  $G$  acts as  $\pm$ -identity on  $P$  (conjugates a minimal rotation to a minimal rotation). Let  $G_0$  be the subgroup of index one or two of  $G$  acting trivially on  $P$ , and let  $Q_0$  be its image in  $Q$ . Then  $G_0 \cong P \times Q_0$  is an abelian subgroup of index at most two in  $G$ , so we are done in this case.

Now suppose that the fixed point set  $F$  of  $P$  has dimension three (and also codimension three). This implies that  $p = 2$  since, if  $p$  is odd, by an inductive argument on the  $p$ -group  $P$ , its fixed point set  $F$  has even codimension. Considering the action of  $P$  on a 3-ball transverse to  $F$  in some point,  $P$  is a subgroup of the orthogonal group  $\text{SO}(3)$  and hence isomorphic to a cyclic or dihedral 2-group.

If  $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  is isomorphic to the Klein 4-group then the subgroup  $G_0$  of  $G$  acting by conjugation trivially on  $P$  has index at most three in  $G$  (since  $Q$  is a  $q$ -group of odd order) and is abelian, so we are done. If  $P$  is a cyclic 2-group then its automorphism group is also a 2-group; since  $Q$  has odd order,  $G$  acts by conjugation trivially on  $P$ , so  $G$  is abelian and we are done. If  $P$  is a dihedral 2-group of order at least eight then it

has a cyclic characteristic subgroup  $P_0$  of index two, so  $G$  has a subgroup  $G_0 = P_0 \rtimes Q$  of index two; by the previous case,  $G_0$  is abelian and we are done.

Suppose next that  $F$  has dimension two. By Smith fixed point theory,  $F$  is an acyclic manifold mod  $p$  (i.e., for homology with coefficients in  $\mathbb{Z}_p$ ). Since  $F$  has dimension two, it is in fact acyclic also for integer coefficients (see [GMZ, proof of Lemma 3]). Then the finite  $q$ -group  $Q$  has a fixed point in  $F$ , and hence  $G$  has a global fixed point. Now  $G$  is isomorphic to a subgroup of  $\mathrm{SO}(6)$ , so we are done by the classical Jordan bound.

The cases that  $F$  has dimension one or zero are similar.

This completes the proof of Theorem 3.

Remark. Considering the next case of  $\mathbb{R}^7$ , if the fixed point set  $F$  of  $P$  has codimension two or three, or if it has dimension at most two, the proof works exactly as before. The case we cannot handle at present is when  $F$  has dimension three (and codimension four). In this case  $P$  is isomorphic to a subgroup of  $\mathrm{SO}(4)$ , e.g. isomorphic to  $\mathbb{Z}_p$  or  $\mathbb{Z}_p \times \mathbb{Z}_p$ , and we don't know how to bound the index of the subgroup of  $G$  (or  $Q$ ) acting trivially on  $P$  (independent of the prime  $p$ ).

## References

- [B] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York 1972
- [C] M.J. Collins, *On Jordan's theorem for complex linear groups*, J. Group Theory 10, 411-423 (2007)
- [GMZ] A. Guazzi, M. Mecchia, B. Zimmermann, *On finite groups acting on acyclic low-dimensional manifolds*, Fund. Math. 215, 203-217 (2011)
- [GZ] A. Guazzi, B. Zimmermann, *On finite simple groups acting on homology spheres*, Monatsh. Math. 169, 371-381 (2013)
- [K] S. Kojima, *Bounding finite groups acting on 3-manifolds*, Math. Proc. Camb. Phil. Soc. 96, 269-281 (1984)
- [MeZ1] M. Mecchia, B. Zimmermann, *On finite simple and nonsolvable groups acting on homology 4-spheres*, Top. Appl. 153, 2933-2942 (2006)
- [MeZ2] M. Mecchia, B. Zimmermann, *On finite groups acting on homology 4-spheres and finite subgroups of  $\mathrm{SO}(5)$* , Top. Appl. 158, 741-747 (2011)
- [MS] W.H. Meeks, P. Scott, *Finite group actions on 3-manifolds*, Invent. math. 86, 287-346 (1986)
- [MR1] I. Mundet i Riera, *Finite groups acting on manifolds without odd cohomology*, arXiv:1310.6565
- [MR2] I. Mundet i Riera, *Finite group actions on 4-manifolds with nonzero Euler characteristic*, arXiv:1312.3149

- [MT] I. Mundet i Riera, A. Turull, *Boosting an analogue of Jordan's theorem for finite groups*, arXiv:1310.6518
- [P1] V.L. Popov, *Finite subgroups of diffeomorphism groups*, arXiv:1310.6548
- [P2] V.L. Popov, *Jordan groups and automorphism groups of algebraic varieties*, arXiv:13007.5522
- [Sc] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15, 401-487 (1983)
- [Se] J.-P. Serre, *Le groupe de Cremona et ses sous-groupes finis*, Sem. Bourbaki 1000, 75-100 (2008)
- [Z1] B. Zimmermann, *On finite groups acting on spheres and finite subgroups of orthogonal groups*, Sib. Electron. Math. Rep. 9, 1 - 12 (2012) (<http://semr.math.nsc.ru>)
- [Z2] B. Zimmermann, *On finite groups acting on a connected sum of 3-manifolds  $S^2 \times S^1$* , arXiv:1202.5427 (to appear in Fund. Math. 2014)